CVXMOD: Convex Optimization in Python

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CVXMOD

- convex optimization modeling layer, in Python
- completely open source, object-oriented toolchain
- form problems easily using basic set of atoms and composition rules from convex analysis
- uses CVXOPT's general nonlinear convex solver (Vandenberghe, Dahl 2005)
- **generate custom C for real-time embedded convex optimization**

Outline

- **modeling languages and disciplined convex programming**
- example: optimal execution
- real-time embedded optimization
- code generation

History

- general purpose optimization modeling systems AMPL, GAMS (1970s), many others...
- systems for SDPs/LMIs (1990s): SDPSOL (Wu, Boyd), LMILAB (Gahinet, Nemirovsky), LMITOOL (El Ghaoui)
- YALMIP (Löfberg 2000–)
- automated convexity checking (Crusius PhD thesis 2002)
- disciplined convex programming (DCP) (Grant, Boyd, Ye 2004)
- CVX (Grant, Boyd, Ye 2005)
- CVXOPT (Dahl, Vandenberghe 2005)
- GGPLAB (Mutapcic, Koh, et al 2006)

Determining convexity—two approaches

- user creates (almost any) model; system attempts to verify convexity
  - hard problem
  - best effort
  - detect convexity with automatic differentiation and interval analysis
    Orban, Fourer 2004 (Dr. AMPL), Nenov, Fylstra, Kolev 2004

- user creates model following a restricted set of rules that ensure convexity
  - model is convex by construction
  - seems more useful to us in engineering
  - user must have some understanding and skill
  - CVX, CVXMOD

Disciplined convex programming

- convex-by-construction method
- expressions appearing in objective and constraints are formed from
  - an extensible set of atoms (functions)
  - a small set of combination rules derived from convex analysis
- rule set is intentionally small; not "as many rules as possible"
- we've found it surprisingly versatile

Example

$$\|Ax - b\|_2$$

- made from variable x, parameters A and b, atom $\|\cdot\|_2$
- expression $Ax - b$ is affine in x
- composite expression $\|Ax - b\|_2$ is convex, positive, non-monotonic
  - could use it in objective, minimize($\|Ax - b\|_2$)
  - could use it in constraint, $\|Ax - b\|_2 \leq 1$
- represent in CVXMOD as `norm2(A*x - b)`

Composition rules

- can combine atoms using valid composition rules, *e.g.*:
  - a convex function of an affine function is convex
  - the negative of a convex function is concave
  - a convex, nondecreasing function of a convex function is convex
  - a concave, nondecreasing function of a concave function is concave
- for convex h, $h(g_1, \ldots, g_k)$ is recognized as convex if, for each i,
  - g_i is affine, or
  - g_i is convex and h is nondecreasing in its ith arg, or
  - g_i is concave and h is nonincreasing in its ith arg
- for concave h, $h(g_1, \ldots, g_k)$ is recognized as concave if, for each i,
  - g_i is affine, or
  - g_i is convex and h is nonincreasing in ith arg, or
  - g_i is concave and h is nondecreasing in ith arg

Valid (recognized) examples

`u`, `v`, `x`, `y` are scalar variables; `X` is a symmetric 3×3 variable

- convex:
  - `norm2(A*x - y) + 0.1*norm1(x)`
  - `maxeig(2*X - 4*eye(3))`
- concave:
  - `min(1 + 2*u, 1 - max(2, v))`
  - `sqrt(v) - 4.55*invpos(u - v)`

Rejected examples

`u`, `v`, `x`, `y` are scalar variables

- neither convex nor concave:
  - `square(x) - square(y)`
  - `norm(A*x - y) - 0.1*norm(x, 1)`

- rejected due to limited DCP ruleset:
  - `sqrt(sum(square(x)))` (is convex; could use `norm(x)`)
  - `norm(x) - 0.1*norm(x)` (is convex; could use `0.9*norm(x)`)

Problem transformation

- DCP makes automatic transformation to convex standard form easy
- based on epigraphical transformations

Transformation example

variables x, y; parameters A, b

$$\|Ax - b\|_\infty \leq 3\log(y)$$

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- introduce variable t_1, to get

$$t_1 = Ax - b$$
$$\|t_1\|_\infty \leq 3\log(y)$$

Transformation example

variables x, y; parameters A, b

$$\|Ax - b\|_\infty \leq 3\log(y)$$

- introduce t_2, to get

$$t_1 = Ax - b$$
$$t_2 \leq 3\log(y)$$
$$-t_2\mathbf{1} \leq t_1 \leq t_2\mathbf{1}$$

Transformation example

variables x, y; parameters A, b

$$\|Ax - b\|_\infty \leq 3\log(y)$$

- lastly, introduce variable t_3, to get

$$\begin{aligned} t_1 &= Ax - b \\ t_2 &\leq 3t_3 \\ -t_2\mathbf{1} \leq t_1 &\leq t_2\mathbf{1} \\ t_3 &\leq \log(y) \end{aligned}$$

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Optimal execution

- execute a sell order for S shares over T time periods
- prices modeled as random walk, plus price decrease from current and previous sales
- maximize expected revenue
- yields (convex) quadratic program

$$
\begin{array}{ll}
\text{maximize} & \bar{p}^T s - s^T Q s \\
\text{subject to} & 0 \leq s \leq S^{\max} \\
& \mathbf{1}^T s = S
\end{array}
$$

- obvious initialization of sales: $s_i = S/T$, $i = 1, \ldots, T$

Specifying problem family in CVXMOD

optimal sell order execution

```
s = optvar('s', T)
S = param('S')
pbar = param('pbar')
prob = problem(maximize(tp(pbar)*s - quadform(s, Q)),
               [0 <= s, s <= Smax, sum(s) == S])
```

(`Q`, `Smax`, `T` have previously defined numerical values)

- describes a problem family, parameterized by `pbar`, `S`
- every symbol (`s`, `pbar`, `prob`, ...) is a (manipulable) Python object

Solving problem instance in CVXMOD

```
pbar.value = linspace(10, 12, 20)
S.value = 100 + 1000*rand()
prob.solve()
```

- `prob` now contains a problem instance
- `prob.solve()`
  - transforms `prob` into standard form
  - calls CVXOPT's solver
  - transforms solution back to original variables
- after solution, access `value()` of any cvxmod object (`value(s)`, `value(s <= Smax)`, `quadform(s, Q)`, ...)

CVXMOD performance

- set $T = 20$ time periods, get problem with 20 variables, 41 constraints
- takes 85 ms to solve (nothing special, Python language overhead)

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Solving problems—two scenarios

- human-in-loop optimization
  - single instance
  - structure recognized and exploited at solve time
  - example: IC design
- embedded optimization
  - no human involved
  - many instances, same structure (same problem family)
  - real-time deadlines
  - example: model predictive control
- not a strict distinction

Embedded optimization

- compile time almost doesn't matter
- detect and exploit structure once, at compile time
- solve time is critical: $\mathcal{O}(\text{ms})$ or $\mathcal{O}(\mu\text{s})$, even $\mathcal{O}(\text{ns})$
- relaxed accuracy requirements
- can exploit clever initializations (including warm start)
- currently done by custom code development

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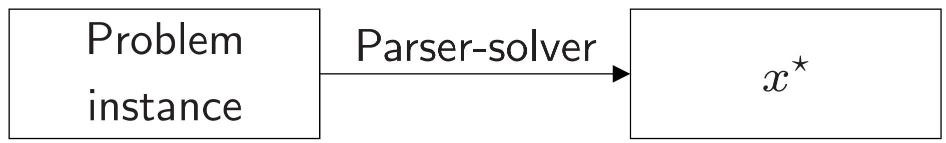
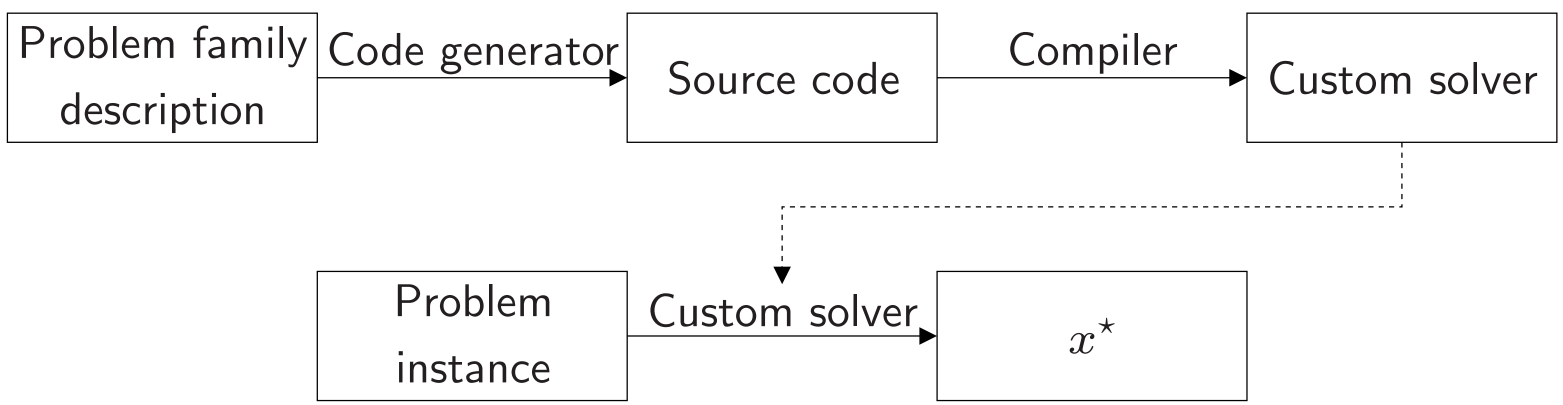
Two scenarios

- human-in-loop



- code generation



Code generation in CVXMOD

- preliminary implementation (not yet released)
- generates C source code from CVXMOD problem family specification
- solver uses simple primal barrier method
- at code generation time
  - analyzes sparsity
  - determines memory arrangements
- search direction computation uses no libraries (faster) or CHOLMOD (smaller code)

Generating C code

```
prob.codegen()
```

produces:

```
solver.c   template.c   README   doc.tex   Makefile   test.c
```

- produces custom C solver and documentation
- provides skeleton for integrating with other code

Template code

```
#include solver.h
int main(int argc, char **argv) {
    CG_params params = initparams();
    CG_vars vars = initvars();
    CG_work work = initwork(vars);

    for (;;) { // Control loop.
        // Get new parameter values.

        status = solve(params, vars, work);

        // Test status, export variables, etc.
    }
}
```

C solver performance

- optimal execution with $T = 20$ steps
- code generation time: 1.1 s, compilation time: 2.3 s
- solve time with C solver: 130 μs (650$\times$ speedup)

Using CVXMOD

- prototype, test, simulate model in Python
- generate C solver source
- embed in application

Summary

- CVXMOD as a modeling language using disciplined convex programming
- real-time embedded optimization ($\mathcal{O}(\mu\text{s})$)
- CVXMOD as a **code generator** for real-time embedded optimization